The separability of tripartite Gaussian states with amplification and amplitude damping

2008 J. Phys. A: Math. Theor. 41 155301

(http://iopscience.iop.org/1751-8121/41/15/155301)

J. Phys. A: Math. Theor. **41** (2008) 155301 (10pp) doi:10.1088/1751-8113/41/15/155301

The separability of tripartite Gaussian states with amplification and amplitude damping

Xiao-Yu Chen

College of Information and Electronic Engineering, Zhejiang Gongshang University, Hangzhou, 310018, People's Republic of China

Received 12 January 2008, in final form 27 February 2008
Published 2 April 2008
Online at stacks.iop.org/JPhysA/41/155301

Abstract
The most general evolution solution of multi-mode continuous variable states is given when it undergoes amplification, amplitude damping and thermal noise. In the case of some kind of initial Gaussian state totally symmetrically interacting with the environment, the conditions for full separability and full entanglement of the final tripartite three-mode Gaussian state are worked out.

PACS numbers: 03.65.Yz, 42.50.Dv, 42.50.Lc

1. Introduction

Quantum entanglement of continuous variables (CV) [1–3] has received much attention recently. Deterministic teleportation schemes [4–8], quantum key distribution protocols [9], entanglement swapping [6, 10, 11], dense coding [12], quantum state storage [13] and quantum computation [14] processes in quantum optical settings have been implemented. Among all quantum CV states, quantum Gaussian states are well studied. Theoretically, almost all the results about the separability and entanglement measures are first done on Gaussian systems, then the results are extended to non-Gaussian systems if possible. Experimentally, essentially all the experimentally realizable CV states are Gaussian. Former works are mainly on the properties of bipartite systems. The study of CV multipartite entanglement was initiated in [7, 10], where a scheme was suggested to create pure CV N-party entanglement using squeezed light and $N - 1$ beam splitters. In the practical situation, such a pure multipartite entanglement state will evolve to a mixed state, due to the interaction with the environment. The effect of the environment on the quantum state is called decoherence. Amplitude damping and thermal noise are two important kinds of decoherence. To overcome the loss in amplitude, amplification may be used. We will investigate the state evolution of the quantum CV state in the environment of amplitude damping, thermal noise and amplification. Of all the multipartite CV states, the tri-mode entangled state is the simplest one, and a complete classification of tri-mode entanglement was obtained, a directly computable criterion that allows us to determine to which class a given state belongs [15, 16]. We will analyze the separability of tripartite

Gaussian states in the presence of amplitude damping, parametric amplification and noise which are symmetric among the modes, based on our former works on the corresponding problem of bipartite CV systems [17, 18].

2. Time evolution of the characteristic function

The density matrix obeys the following master equation [19–21],

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[H,\rho] + \mathcal{L}\rho, \tag{1}$$

with the quadratic Hamiltonian

$$H = \hbar\sum_{jk}\frac{\mathrm{i}}{2}\left(\eta_{jk}a_j^\dagger a_k^\dagger - \eta_{jk}^* a_j a_k\right), \tag{2}$$

where η_{jk} are the entries of complex symmetric matrix $\boldsymbol{\eta}$. In the single-mode case, this Hamiltonian describes two-photon down-conversion from an undepleted (classical) pump [21]. The full multi-mode model describes quasi-particle excitation in a BEC within the Bogoliubov approximation [22]. This item represents the parametric amplifier. While the amplitude damping is described by $\mathcal{L}$,

$$\mathcal{L}\rho = \sum_j \frac{\Gamma_j}{2}\{(n_j+1)L[a_j]\rho + n_j L\left[a_j^\dagger\right]\rho, \tag{3}$$

where the Lindblad super-operator is defined as $L[\widehat{o}]\rho \equiv 2\widehat{o}\rho\widehat{o}^\dagger - \widehat{o}^\dagger\widehat{o}\rho - \rho\widehat{o}^\dagger\widehat{o}$. Γ_j is the damping coefficient and n_j is the noise of the jth mode.

We now transform the density operator master equation to the diffusion equation of the characteristic function. Any quantum state can be equivalently specified by its characteristic function. Every operator $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ is completely determined by its characteristic function $\chi_\mathcal{A} := \mathrm{tr}[\mathcal{A}\mathcal{D}(\mu)]$ [23], where $\mathcal{D}(\mu) = \exp(\mu a^\dagger - \mu^* a)$ is the displacement operator, with $\mu = [\mu_1, \mu_2, \ldots, \mu_s]$, $a = [a_1, a_2, \ldots, a_s]^T$ and the total number of modes being s. It follows that $\mathcal{A}$ may be written in terms of $\chi_\mathcal{A}$ as [24]: $\mathcal{A} = \int\left[\prod_j \frac{\mathrm{d}^2\mu_j}{\pi}\right]\chi_\mathcal{A}(\mu)\mathcal{D}(-\mu)$. The density matrix ρ can be expressed with its characteristic function χ. $\chi = \mathrm{tr}[\rho\mathcal{D}(\mu)]$. Multiplying $\mathcal{D}(\mu)$ to the master equation then taking the trace, the master equation of the density operator will be transformed to the diffusion equation of the characteristic function. It should be noted that the complex parameters μ_j are not a function of time, thus $\frac{\partial\chi}{\partial t} = \mathrm{tr}\left[\frac{\partial\rho}{\partial t}\mathcal{D}(\mu)\right]$; the parametric amplification part in the form of the characteristic function will be [21]

$$\frac{1}{2}\mathrm{tr}\left\{\sum_{jk}\left[\eta_{jk}a_j^\dagger a_k^\dagger - \eta_{jk}^* a_j a_k, \rho\right]D(\mu)\right\} = -\sum_{jk}\left(\eta_{jk}\mu_j^*\frac{\partial\chi}{\partial\mu_k} + \eta_{jk}^*\mu_j\frac{\partial\chi}{\partial\mu_k^*}\right). \tag{4}$$

The master equation can be transformed into the diffusion equation of the characteristic function; it is [17]

$$\frac{\partial\chi}{\partial t} = -\sum_{jk}\left(\eta_{jk}\mu_j^*\frac{\partial\chi}{\partial\mu_k} + \eta_{jk}^*\mu_j\frac{\partial\chi}{\partial\mu_k^*}\right) - \frac{1}{2}\sum_j\Gamma_j\left\{|\mu_j|\frac{\partial\chi}{\partial|\mu_j|} + (2n_j+1)|\mu_j|^2)\chi\right\}. \tag{5}$$

The density operator can be expressed with canonical operators $X = \frac{1}{\sqrt{2}}(a + a^\dagger)$, $P = \frac{\mathrm{i}}{\sqrt{2}}(a^\dagger - a)$ (the frequencies of the modes are omitted here) as well. Then the characteristic function can be rewritten with real parameters $x = [x_1, x_2, \ldots, x_s]$, $p = [p_1, p_2, \ldots, p_s]$.

Let $\mathcal{D}(\mu) = \exp(\mu a^{\dagger} - \mu^* a) = \exp[\mathrm{i}(xX + pP)]$, we have $\chi(\mu) = \chi(x, p)$, with $x = -\frac{\mathrm{i}}{\sqrt{2}}(\mu - \mu^*)$, $p = -\frac{1}{\sqrt{2}}(\mu + \mu^*)$. The diffusion equation of $\chi(x, p)$ is

$$\frac{\partial \chi}{\partial t} = [x, p]\mathbf{W}\left[\frac{\partial \chi}{\partial x}, \frac{\partial \chi}{\partial p}\right]^T - \frac{1}{2}[x, p]\left[\mathbf{\Gamma}\left(\mathbf{n} + \frac{\mathbf{I}}{2}\right) \oplus \mathbf{\Gamma}\left(\mathbf{n} + \frac{\mathbf{I}}{2}\right)\right][x, p]^T \chi, \tag{6}$$

with

$$\mathbf{W} = \begin{bmatrix} \boldsymbol{\eta}_R - \frac{\mathbf{\Gamma}}{2}, & \boldsymbol{\eta}_I \\ \boldsymbol{\eta}_I, & -\boldsymbol{\eta}_R - \frac{\mathbf{\Gamma}}{2} \end{bmatrix}, \tag{7}$$

where the real matrices $\boldsymbol{\eta}_R$ and $\boldsymbol{\eta}_I$ are the real and imaginary parts of the matrix $\boldsymbol{\eta}$, and $\mathbf{\Gamma} = \mathrm{diag}\{\Gamma_1, \Gamma_2, \ldots, \Gamma_s\}$, $\mathbf{n} = \mathrm{diag}\{n_1, n_2, \ldots, n_s\}$, $\mathbf{I}$ is the s-dimensional identity matrix.

Suppose the solution to the diffusion equation (6) is

$$\chi(x, p, t) = \chi(x', p', 0) \exp\left[\tfrac{1}{4}(x', p')\alpha_0(x', p')^T - \tfrac{1}{4}(x, p)\alpha_0(x, p)^T\right], \tag{8}$$

where $x' = x\mathbf{M}_1 + p\mathbf{M}_2$, $p' = x\mathbf{M}_3 + p\mathbf{M}_4$, with $\mathbf{M}_j$ being time varying real matrices, α_0 a constant matrix. We may denote $\mathbf{M} = \left[\begin{smallmatrix} \mathbf{M}_1 & \mathbf{M}_2 \\ \mathbf{M}_3 & \mathbf{M}_4 \end{smallmatrix}\right]$, then $\mathbf{M}$ is the solution of the following matrix equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{M} = \mathbf{W}\mathbf{M}, \tag{9}$$

and

$$\alpha_0 = -\mathbf{W}^{-1}\left[\mathbf{\Gamma}\left(\mathbf{n} + \frac{\mathbf{I}}{2}\right) \oplus \mathbf{\Gamma}\left(\mathbf{n} + \frac{\mathbf{I}}{2}\right)\right]. \tag{10}$$

The solution of equation (9) is

$$\mathbf{M} = \exp(\mathbf{W}t) \cdot \mathbf{M}(0). \tag{11}$$

The initial conditions of $\mathbf{M}_j$ are $\mathbf{M}_1(0) = \mathbf{M}_4(0) = \mathbf{I}_s$ and $\mathbf{M}_2(0) = \mathbf{M}_3(0) = \mathbf{0}$, thus the solutions are

$$\mathbf{M} = \exp(\mathbf{W}t) \cdot \mathbf{I}_{2s}. \tag{12}$$

The solutions are especially simple when $\boldsymbol{\eta}$ is real. They are $\mathbf{M}_1 = \exp\left(\boldsymbol{\eta}t - \frac{\mathbf{\Gamma}t}{2}\right)$, $\mathbf{M}_2 = \mathbf{0}$, $\mathbf{M}_3 = \mathbf{0}$, $\mathbf{M}_4 = \exp\left(-\boldsymbol{\eta}t - \frac{\mathbf{\Gamma}t}{2}\right)$.

3. The separability criterion of the tripartite Gaussian state

The separability problem of the three-mode Gaussian state was perfectly solved [15]. The three-mode Gaussian states were classified as five different entangled classes [15]. But the states in this paper can be classified as three different entangled classes: fully inseparable states, biseparable states, and fully separable states. Following the notation of [15], the correlation matrix (CM) is denoted as γ, the partial transposition is denoted as $\Lambda_j (j = 1, 2, 3)$. If the canonical operators are arranged in the order of $X_1, P_1, X_2, P_2, X_3, P_3$, one has $\Lambda_1 = \mathrm{diag}\{1, -1, 1, 1, 1, 1\}$, $\Lambda_2 = \mathrm{diag}\{1, 1, 1, -1, 1, 1\}\Lambda_3 = \mathrm{diag}\{1, 1, 1, 1, 1, -1\}$. The partially transposed CM will be $\widetilde{\gamma}_j = \Lambda_j \gamma \Lambda_j$. Denote

$$J_n = \bigoplus_{i=1}^{n} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{13}$$

then the criterion for the fully inseparable state is

$$\widetilde{\gamma}_j \ngeq \mathrm{i}J_3, \qquad \text{for all} \quad j = 1, 2, 3. \tag{14}$$

Because of the symmetry of the tri-mode state, the criterion can be simplified to, for example, $\widetilde{\gamma}_A \ngeqslant \mathrm{i}J$.

While for $\widetilde{\gamma}_j \geqslant \mathrm{i}J_3$, $(j = 1, 2, 3)$, the state will be a positive partial transpose (PPT) tri-mode state, and it can be biseparable or fully separable. The criterion to distinguish the biseparable and fully separable states is [15] as follows. The CM γ of the PPT tri-mode state can be written as

$$\gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \tag{15}$$

where A is a 2×2 CM for the first mode, whereas B is a 4×4 CM for the other two modes. Define the matrices K and $\widetilde{K}$ as

$$K \equiv A - C\frac{1}{B - \mathrm{i}J_2}C^T, \qquad \widetilde{K} \equiv A - C\frac{1}{B - \mathrm{i}\widetilde{J}_2}C^T, \tag{16}$$

where $\widetilde{J}_2 = J_1 \oplus (-J_1)$ is the partially transposed J_2.

The condition of the PPT tri-mode state being fully separable is that if and only if there exists a point $(y, z) \in R^2$ fulfilling the following inequality:

$$\min\{\mathrm{tr}K, \mathrm{tr}\widetilde{K}\} \geqslant 2x, \tag{17}$$

$$\det K + 1 + L^T(y,\ z)^T \geqslant x \cdot\ \mathrm{tr}K, \tag{18}$$

$$\det \widetilde{K} + 1 + \widetilde{L}^T(y,\ z)^T \geqslant x \cdot\ \mathrm{tr}\widetilde{K}, \tag{19}$$

where $x = \sqrt{1 + y^2 + z^2}$ (note that x, z in this section should not be confused with that used in the other sections) and $L = (u - w, 2\,\mathrm{Re}(v))$, $\widetilde{L} = (\widetilde{u} - \widetilde{w}, 2\,\mathrm{Re}(\widetilde{v}))$ if K and $\widetilde{K}$ are written as

$$K = \begin{pmatrix} u & v \\ v^* & w \end{pmatrix}, \qquad \widetilde{K} = \begin{pmatrix} \widetilde{u} & \widetilde{v} \\ \widetilde{v}^* & \widetilde{w} \end{pmatrix}. \tag{20}$$

Inequality (17) restricts (y, z) to a circular disc $\mathcal{C}$, while inequalities (18) and (19) describe ellipses $\mathcal{E}$ and $\mathcal{E}'$ respectively. The existence of the point (y, z) then turns out to be the intersection of the ellipses $\mathcal{E}$ and $\mathcal{E}'$ and the circular disc $\mathcal{C}$.

The intersection of the ellipses $\mathcal{E}$ and $\mathcal{E}'$ is a range in the yz plane which is bounded by the elliptic curves $\partial\mathcal{E}$ and $\partial\mathcal{E}'$. In the cases considered in this paper, Re $(v) = 0$, Re $(\widetilde{v}) = 0$, the two elliptic curves $\partial\mathcal{E}$ and $\partial\mathcal{E}'$ are described by

$$\det K + 1 + (u - w)y = (u + w)x, \tag{21}$$

$$\det \widetilde{K} + 1 + (\widetilde{u} - \widetilde{w})y = (\widetilde{u} + \widetilde{w})x, \tag{22}$$

$\partial\mathcal{E}$ and $\partial\mathcal{E}'$ are centered at the y-axis of the yz plane. The intersection of $\partial\mathcal{E}$ and $\partial\mathcal{E}'$ is the solution of these two equations as far as

$$x \geqslant \sqrt{1 + y^2}. \tag{23}$$

Thus the condition of the existence of $(\partial\mathcal{E}) \cap (\partial\mathcal{E}')$ is obtained.

Two situations of the intersection of the ellipses $\mathcal{E}$ and $\mathcal{E}'$ and the circular disc $\mathcal{C}$ should be considered. The first is $((\partial\mathcal{E}) \cap (\partial\mathcal{E}')) \subseteq \mathcal{C}$; in this case, the fully separability condition is determined by (21), (22) and (23). The second is $((\partial\mathcal{E}) \cap (\partial\mathcal{E}')) \subsetneq \mathcal{C}$; in this case, we should consider that if one of the tops of $\mathcal{E} \cap \mathcal{E}'$ is contained in $\mathcal{C}$ or not, the two tops are determined by equations (21) and (22) separately by setting $x = \sqrt{1 + y^2}$.

4. The symmetric amplification and damping of the tripartite Gaussian state

4.1. Symmetric evolution of the tripartite state

We consider the totally symmetric amplification and amplitude damping among all three modes, that is

$$\boldsymbol{\eta} = \eta_0\mathbf{I}_3 + \eta_1\mathbf{S}_3, \tag{24}$$

with the matrix $\mathbf{S}$ having its entries $S_{ij} = 1$ for $i \neq j$ and $S_{ij} = 0$ for $i = j (i, j = 1, 2, 3)$; $\boldsymbol{\Gamma} = \Gamma\mathbf{I}_3, \mathbf{n} = n\mathbf{I}_3$, and $\boldsymbol{\eta}_R = \eta_{0R}\,\mathbf{I}_3 + \eta_{1R}\,\mathbf{S}_3, \boldsymbol{\eta}_I = \eta_{0I}\,\mathbf{I}_3 + \eta_{1I}\mathbf{S}_3$. To obtain the time evolution of the state, we need to consider the matrix $\mathbf{W}$ defined in (7). The exponential expression $\exp(\mathbf{W}t)$ can be simplified to $\exp(\mathbf{W}t) = \mathrm{e}^{-\Gamma t/2}\exp\left(\left[\begin{smallmatrix}\boldsymbol{\eta}_R, & \boldsymbol{\eta}_I\\ \boldsymbol{\eta}_I, & -\boldsymbol{\eta}_R\end{smallmatrix}\right]t\right)$. Note that $\boldsymbol{\eta}_R$ and $\boldsymbol{\eta}_I$ commutate with each other; we have

$$\begin{bmatrix}\boldsymbol{\eta}_R, & \boldsymbol{\eta}_I\\ \boldsymbol{\eta}_I, & -\boldsymbol{\eta}_R\end{bmatrix}^2 = \begin{bmatrix}\boldsymbol{\eta}_R^2 + \boldsymbol{\eta}_I^2, & 0\\ 0, & \boldsymbol{\eta}_R^2 + \boldsymbol{\eta}_I^2\end{bmatrix}. \tag{25}$$

For a matrix with the form of (24), we can rewrite it as $\boldsymbol{\eta} = \eta_0\mathbf{I}_3 + \eta_1\mathbf{S}_3 = (\eta_0 + 2\eta_1)\mathbf{P}_1 + (\eta_0 - \eta_1)\mathbf{P}_2$, with $\mathbf{P}_1 = \frac{1}{3}(\mathbf{I}_3 + \mathbf{S}_3), \mathbf{P}_2 = \frac{1}{3}(2\mathbf{I}_3 - \mathbf{S}_3)$. We have $\mathbf{P}_1^2 = \mathbf{P}_1, \mathbf{P}_2^2 = \mathbf{P}_2, \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1 = \mathbf{0}$. Thus $\boldsymbol{\eta}_R^2 + \boldsymbol{\eta}_I^2 = A^2\mathbf{P}_1 + B^2\mathbf{P}_2$, with $A = \sqrt{(\eta_{0R} + 2\eta_{1R})^2 + (\eta_{0I} + 2\eta_{1I})^2}$, $B = \sqrt{(\eta_{0R} - \eta_{1R})^2 + (\eta_{0I} - \eta_{1I})^2}$. We obtain

$$\begin{aligned}\exp(\mathbf{W}t) = \mathrm{e}^{-\Gamma t/2}\{&[\cosh(At)\mathbf{P}_1 + \cosh(Bt)\mathbf{P}_2] \oplus [\cosh(At)\mathbf{P}_1 + \cosh(Bt)\mathbf{P}_2]\\ &+ \begin{bmatrix}\boldsymbol{\eta}_R, & \boldsymbol{\eta}_I\\ \boldsymbol{\eta}_I, & -\boldsymbol{\eta}_R\end{bmatrix}\left[\frac{\sinh(At)}{A}\mathbf{P}_1 + \frac{\sinh(Bt)}{B}\mathbf{P}_2\right]\oplus\left[\frac{\sinh(At)}{A}\mathbf{P}_1 + \frac{\sinh(Bt)}{B}\mathbf{P}_2\right].\end{aligned} \tag{26}$$

The time-dependent matrix $\mathbf{M}$ is

$$\mathbf{M} = \begin{bmatrix}b_1\mathbf{P}_1 + b_2\mathbf{P}_2, & b_3\mathbf{P}_1 + b_4\mathbf{P}_2\\ b_3\mathbf{P}_1 + b_4\mathbf{P}_2, & b_5\mathbf{P}_1 + b_6\mathbf{P}_2\end{bmatrix}, \tag{27}$$

with $b_1 = \mathrm{e}^{-\Gamma t/2}\left[\cosh(At) + \frac{\eta_{0R}+2\eta_{1R}}{A}\sinh(At)\right]$, $b_2 = \mathrm{e}^{-\Gamma t/2}\left[\cosh(Bt) + \frac{\eta_{0R}-\eta_{1R}}{B}\sinh(Bt)\right]$, $b_3 = \frac{\eta_{0I}+2\eta_{1I}}{A}\sinh(At)\,\mathrm{e}^{-\Gamma t/2}$, $b_4 = \frac{\eta_{0I}-\eta_{1I}}{B}\sinh(Bt)\,\mathrm{e}^{-\Gamma t/2}$, $b_5 = \mathrm{e}^{-\Gamma t/2}\left[\cosh(At) - \frac{\eta_{0R}+2\eta_{1R}}{A}\sinh(At)\right]$, $b_6 = \mathrm{e}^{-\Gamma t/2}\left[\cosh(Bt) - \frac{\eta_{0R}-\eta_{1R}}{B}\sinh(Bt)\right]$. The inverse of the matrix $\mathbf{W}$ can be obtained as follows: all the four blocks of $\mathbf{W}$ commutate with each other, so they can be diagonalized simultaneously. The orthogonal matrix that diagonalizes all the four blocks is

$$\mathbf{U} = \frac{1}{\sqrt{6}}\begin{bmatrix}\sqrt{2} & \sqrt{2} & \sqrt{2}\\ 0 & \sqrt{3} & -\sqrt{3}\\ -2 & 1 & 1\end{bmatrix}. \tag{28}$$

The block diagonalized matrix $(\mathbf{U}\oplus\mathbf{U})\mathbf{W}(\mathbf{U}\oplus\mathbf{U})^T$ can easily be inverted. We have

$$\mathbf{W}^{-1} = \frac{-1}{\Gamma\left(n + \frac{1}{2}\right)}\begin{bmatrix}d_1\mathbf{P}_1 + d_2\mathbf{P}_2, & d_3\mathbf{P}_1 + d_4\mathbf{P}_2\\ d_3\mathbf{P}_1 + d_4\mathbf{P}_2, & d_5\mathbf{P}_1 + d_6\mathbf{P}_2\end{bmatrix}, \tag{29}$$

with $(d_1, d_2, d_3, d_4, d_5, d_6) = \Gamma\left(n + \frac{1}{2}\right)\left[\frac{\Gamma/2+\eta_{0R}+2\eta_{1R}}{(\Gamma/2)^2-A^2}, \frac{\Gamma/2+\eta_{0R}-\eta_{1R}}{(\Gamma/2)^2-B^2}, \frac{\eta_{0I}+2\eta_{1I}}{(\Gamma/2)^2-A^2}, \frac{\eta_{0I}-\eta_{1I}}{(\Gamma/2)^2-B^2}, \frac{\Gamma/2-\eta_{0R}-2\eta_{1R}}{(\Gamma/2)^2-A^2}, \frac{\Gamma/2-\eta_{0R}+\eta_{1R}}{(\Gamma/2)^2-B^2}\right]$. The constant matrix is

$$\alpha_0 = \begin{bmatrix}d_1\mathbf{P}_1 + d_2\mathbf{P}_2, & d_3\mathbf{P}_1 + d_4\mathbf{P}_2\\ d_3\mathbf{P}_1 + d_4\mathbf{P}_2, & d_5\mathbf{P}_1 + d_6\mathbf{P}_2\end{bmatrix}.$$

$\alpha_0 = -\Gamma\left(n + \frac{1}{2}\right)\mathbf{W}^{-1}$.

4.2. Gaussian initial tripartite state

If the initial state is Gaussian, then the time-dependent state will be Gaussian. For an initial Gaussian state with the characteristic function $\chi(x,p,0) = \exp\left[-\frac{1}{4}(x,p)\alpha(0)(x,p)^T\right]$, the evolution of the characteristic function is $\chi(x,p,t) = \exp\left[-\frac{1}{4}(x,p)\alpha(t)(x,p)^T\right]$. The time-dependent CM (in the order of canonical operators $X_1, X_2, X_3, P_1, P_2, P_3$ or parameters $x_1, x_2, x_3, p_1, p_2, p_3$) is

$$\alpha(t) = \mathbf{M}[\alpha(0) - \alpha_0)]\mathbf{M}^T + \alpha_0. \tag{30}$$

We consider the initial Gaussian state which is symmetric among all the three modes, that is,

$$\alpha(0) = \begin{bmatrix} c_1\mathbf{P}_1 + c_2\mathbf{P}_2, & c_3\mathbf{P}_1 + c_4\mathbf{P}_2 \\ c_3\mathbf{P}_1 + c_4\mathbf{P}_2, & c_5\mathbf{P}_1 + c_6\mathbf{P}_2 \end{bmatrix}. \tag{31}$$

The parameters c_i should be so chosen that the initial state is a quantum state. The matrix $\alpha(t)$ has the same matrix structure as $\alpha(0)$; we denote its corresponding coefficients e_i. Then

$$e_1 = b_1^2(c_1 - d_1) + 2b_1b_3(c_3 - d_3) + b_3^2(c_5 - d_5) + d_1. \tag{32}$$

$$e_3 = b_1b_3(c_1 - d_1) + \left(b_1b_5 + b_3^2\right)(c_3 - d_3) + b_3b_5(c_5 - d_5) + d_3. \tag{33}$$

$$e_5 = b_3^2(c_1 - d_1) + 2b_3b_5(c_3 - d_3) + b_5^2(c_5 - d_5) + d_5. \tag{34}$$

The substitution $1 \to 2, 3 \to 4, 5 \to 6$ into all subscripts of equations (32)–(34) will lead to e_2, e_4, e_6.

The separability criterion is expressed with the CM γ which is arranged in the order of modes, that is, $\chi(x,p,t) = \exp\left[-\frac{1}{4}z\gamma(t)z^T\right]$, with $z = [x_1, p_1, x_2, p_2, x_3, p_3]$. After the rearrangement of the order of x_i and p_i, we have

$$\gamma(t) = \begin{bmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_2 \\ \mathbf{E}_2 & \mathbf{E}_1 & \mathbf{E}_2 \\ \mathbf{E}_2 & \mathbf{E}_2 & \mathbf{E}_1 \end{bmatrix}, \tag{35}$$

with $\mathbf{E}_1 = \left[\begin{smallmatrix} e_1' & e_3' \\ e_3' & e_5' \end{smallmatrix}\right]$, $\mathbf{E}_2 = \left[\begin{smallmatrix} e_2' & e_4' \\ e_4' & e_6' \end{smallmatrix}\right]$ and $e'_{1(3,5)} = \frac{1}{3}(e_{1(3,5)} + 2e_{2(4,6)})$, $e'_{2(4,6)} = \frac{1}{3}(e_{1(3,5)} - e_{2(4,6)})$. The inseparability criteria of the evolved state can be worked out with matrix algebra, but they are still too complicated to be explicitly expressed. In the following section, we will consider the simple case of real parameter amplification.

5. Real amplification with proper Gaussian input

For the real amplification symmetric system, $\boldsymbol{\eta} = \boldsymbol{\eta}_R$, we have $\mathbf{M}_1 = \mathrm{e}^{-\Gamma t/2}[\mathrm{e}^{(\eta_0+2\eta_1)t}\,\mathbf{P}_1 + \mathrm{e}^{(\eta_0-\eta_1)t}\,\mathbf{P}_2]$, $\mathbf{M}_2 = \mathbf{M}_3 = \mathbf{0}$, $\mathbf{M}_4 = \mathrm{e}^{-\Gamma t/2}[\mathrm{e}^{-(\eta_0+2\eta_1)t}\,\mathbf{P}_1 + \mathrm{e}^{-(\eta_0-\eta_1)t}\,\mathbf{P}_2]$; and $\alpha_0 = (2n+1)\left[\left(\frac{\Gamma/2}{\Gamma/2-\eta_0-2\eta_1}\mathbf{P}_1 + \frac{\Gamma/2}{\Gamma/2-\eta_0+\eta_1}\mathbf{P}_2\right) \oplus \left(\frac{\Gamma/2}{\Gamma/2+\eta_0+2\eta_1}\mathbf{P}_1 + \frac{\Gamma/2}{\Gamma/2+\eta_0-\eta_1}\mathbf{P}_2\right)\right]$. If the initial symmetric Gaussian state has the CM

$$\alpha(0) = (c_1\mathbf{P}_1 + c_2\mathbf{P}_2) \oplus (c_5\mathbf{P}_1 + c_6\mathbf{P}_2), \tag{36}$$

then $\alpha(t) = (e_1\mathbf{P}_1 + e_2\mathbf{P}_2) \oplus (e_5\mathbf{P}_1 + e_6\mathbf{P}_2)$, with

$$e_{1,5} = \mathrm{e}^{\pm 2(\eta_0+2\eta_1)t-\Gamma t}c_{1,5} + \frac{\Gamma(2n+1)}{\Gamma \mp 2(\eta_0+2\eta_1)}(1 - \mathrm{e}^{\pm 2(\eta_0+2\eta_1)t-\Gamma t}). \tag{37}$$

$$e_{2,6} = \mathrm{e}^{\pm 2(\eta_0-\eta_1)t-\Gamma t}c_{2,6} + \frac{\Gamma(2n+1)}{\Gamma \mp 2(\eta_0-\eta_1)}(1 - \mathrm{e}^{\pm 2(\eta_0-\eta_1)t-\Gamma t}). \tag{38}$$

5.1. The biseparable conditions

The biseparable condition $\widetilde{\gamma}_A \geqslant \mathrm{i}J$ can be expressed with the relation on e_1', e_2', e_5', e_6'. It is more apparent to be expressed with e_1, e_2, e_5, e_6. A direct calculation leads to the biseparable condition

$$[1-(e_1e_5+8e_2e_5+8e_1e_6+e_2e_6)/9+e_1e_2e_5e_6][e_2e_6-1]\geqslant 0. \tag{39}$$

Denote $\zeta_0 = 2(\eta_0+2\eta_1)/\Gamma$, $\zeta_1 = 2(\eta_0-\eta_1)/\Gamma$, in the case of $\max\{|\zeta_0|, |\zeta_1|\} < 1$, when $t\to\infty$, we have $e_{1,5} = \frac{2n+1}{1\mp\zeta_0}$, $e_{2,6} = \frac{2n+1}{1\mp\zeta_1}$ and $e_2e_6 = \frac{(2n+1)^2}{1-\zeta_1^2} > 1$. All information of the initial state is damped at the last. The solution of inequality (39) is

$$(2n+1)^2 \geqslant 1+\frac{4}{\Gamma^2}\left(-\eta_0^2-\eta_0\eta_1+\frac{3}{2}\eta_1^2+\frac{1}{2}|\eta_1|\sqrt{8\Gamma^2+4\eta_0^2+4\eta_0\eta_1-7\eta_1^2}\right). \tag{40}$$

In the case of $\zeta_0 > 1, |\zeta_1| < 1$. When $t\to\infty$, we have $e_1\to+\infty$, $e_5 = \frac{2n+1}{1+\zeta_0}$, $e_{2,6} = \frac{2n+1}{1\mp\zeta_1}$ (note that if $e_1\to-\infty$, the state should be non-physical). The solution of inequality (39) is

$$(2n+1)^2 \geqslant (1-\zeta_1)(1+\zeta_0)-\tfrac{1}{9}(1-\zeta_1)(\zeta_0-\zeta_1). \tag{41}$$

The second situation we should consider is $\zeta_0 > 1, \zeta_1 < -1$; we have $e_1\to+\infty$, $e_6\to+\infty$, $e_2 = \frac{2n+1}{1+\zeta_0}$, $e_5 = \frac{2n+1}{1-\zeta_1}$ at $t\to\infty$, thus the solution of inequality (40) reduces to

$$(2n+1)^2 \geqslant \tfrac{8}{9}(1-\zeta_1)(1+\zeta_0). \tag{42}$$

Similar results can be obtained for other domains of the parameters ζ_0 and ζ_1, They are

$$(2n+1)^2 \geqslant (1-\zeta_1)(1+\zeta_0)-\tfrac{1}{9}(1+\zeta_0)(\zeta_0-\zeta_1), \qquad \text{for} \quad |\zeta_0|<1,\ \zeta_1<-1; \tag{43}$$

$$(2n+1)^2 \geqslant (1+\zeta_1)(1-\zeta_0)-\tfrac{1}{9}(1-\zeta_0)(\zeta_1-\zeta_0), \qquad \text{for} \quad |\zeta_0|<1,\ \zeta_1>1; \tag{44}$$

$$(2n+1)^2 \geqslant (1+\zeta_1)(1-\zeta_0)-\tfrac{1}{9}(1+\zeta_1)(\zeta_1-\zeta_0), \qquad \text{for} \quad |\zeta_1|<1,\ \zeta_0<-1; \tag{45}$$

$$(2n+1)^2 \geqslant \tfrac{8}{9}(1+\zeta_1)(1-\zeta_0), \qquad \text{for} \quad \zeta_1>1, \quad \zeta_0<-1. \tag{46}$$

There are no restrictions to $2n+1$ in the domain of $\zeta_0 > 1, \zeta_1 > 1$ and domain of $\zeta_0 < -1, \zeta_1 < -1$, which means the final states in these domains are always biseparable. The synthesis of equations (40) and (41)–(46) is the biseparable condition for all the parameters. The biseparable condition is shown in figure 1.

5.2. The fully separable conditions

To find the fully separable condition, we need the matrices K and $\widetilde{K}$, which is not difficult to obtain. We solve equations (21), (22) for x and y and insert them into inequality (23). Although the algebra calculation is complicated in the process, the final result is rather simple. Condition (23) turns out to be

$$-(e_1-e_2)(e_5-e_6)(e_1e_6-1)(e_2e_5-1)\geqslant 0. \tag{47}$$

In the case of $\max\{|\zeta_0|, |\zeta_1|\} < 1$, when $t\to\infty$, we have $-(e_1-e_2)(e_5-e_6) = \frac{(\zeta_0-\zeta_1)^2}{(1-\zeta_0^2)(1-\zeta_1^2)} > 0$ (for $\zeta_0\neq\zeta_1$); the condition of fully separability will be

$$(2n+1)^2 \geqslant (1+\zeta_0)(1-\zeta_1), \qquad \text{for} \quad \zeta_0>\zeta_1, \tag{48}$$

$$(2n+1)^2 \geqslant (1-\zeta_0)(1+\zeta_1), \qquad \text{for} \quad \zeta_0<\zeta_1, \tag{49}$$

which can be rewritten as

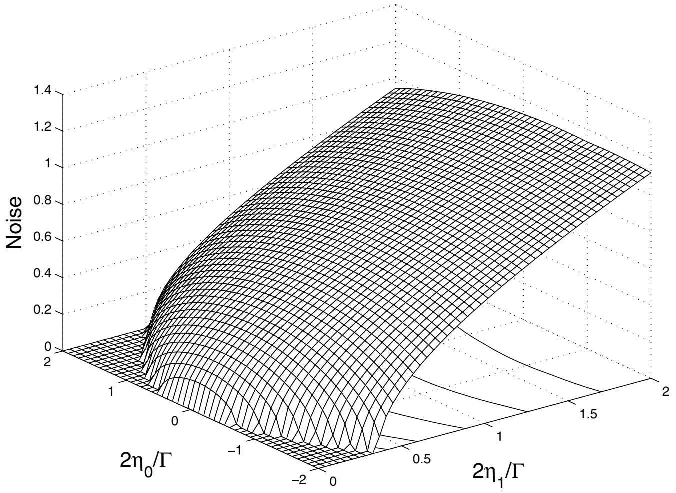

Figure 1. The border of biseparable states and fully entangled states, characterized by the thermal noise n, the ratio of single-mode amplification to amplitude damping, the ratio of inter-mode amplification to amplitude damping.

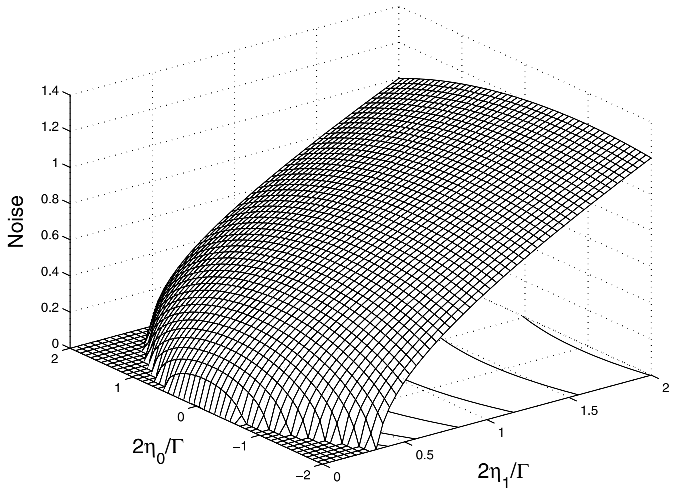

Figure 2. The border of fully separable states and biseparable states, characterized by the thermal noise n, the ratio of single-mode amplification to amplitude damping, the ratio of inter-mode amplification to amplitude damping.

$$(2n+1)^2 \geqslant (1 - 2\eta_0/\Gamma + 2\eta_1/\Gamma)(1 + 2\eta_0/\Gamma + 4\eta_1/\Gamma), \qquad \text{for} \quad \eta_1 > 0, \tag{50}$$

$$(2n+1)^2 \geqslant (1 - 2\eta_0/\Gamma - 4\eta_1/\Gamma)(1 + 2\eta_0/\Gamma - 2\eta_1/\Gamma), \qquad \text{for} \quad \eta_1 < 0, \tag{51}$$

respectively. These are the fully separable conditions of the residue states and are shown in figure 2. Concerning condition (17), we consider the critical situation of $(2n_0 + 1)^2 = (1 + \zeta_0)(1 - \zeta_1)$ for $\zeta_0 > \zeta_1$, condition (17) reduces simply to $\zeta_0 > \zeta_1$, thus the state is fully separable in the critical situation. By physical consideration, a state with $n > n_0$ then is fully

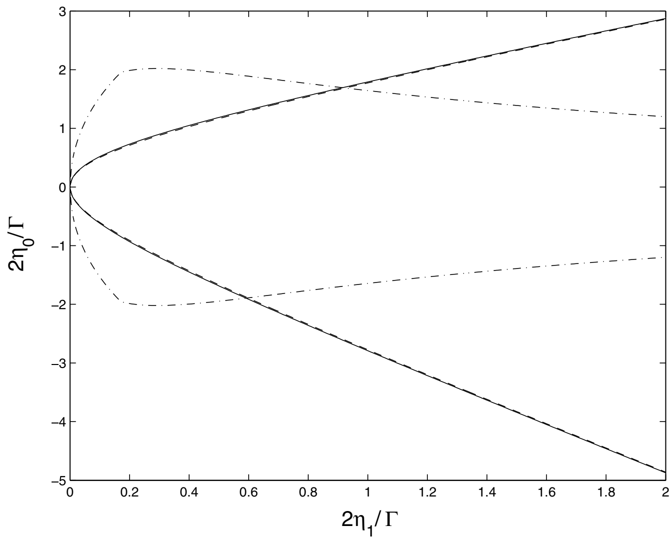

Figure 3. The noiseless situation, the solid line for the border of fully separable and biseparable states, the dashed line for the border of biseparable and fully entangled states. The difference of the two curves is very small, the detail of the difference is amplified by a factor of 100 and is shown by the dashdot line.

separable, because the state will be made more separable by adding the noise. The same conclusion is true for the case of $\zeta_0 < \zeta_1$. Hence the fully separability conditions are given by inequalities (48) and (49) in the situation of $\max\{|\zeta_0|, |\zeta_1|\} < 1$.

Moreover, we can prove that (48) and (49) are also fully separable conditions of the final state ($t \to \infty$) for other situations considered here. We provide the proof of one of the cases here; the other cases can be followed with the same method. The case we considered is $\zeta_0 > 1, |\zeta_1| < 1$; in this case we have $e_1 \to +\infty$, $e_5 = \frac{2n+1}{1+\zeta_0}$, $e_2 = \frac{2n+1}{1-\zeta_1}$, $e_6 = \frac{2n+1}{1+\zeta_1} > 0$, thus $(e_1 - e_2) > 0$, $(e_5 - e_6) < 0$, $e_1 e_6 > 1$ and condition (47) reduces to $e_2 e_5 \geqslant 1$, which leads to (48). Condition (17) can also be fulfilled. A comparison of the figures as well as the formula shows that the fully separable and biseparable conditions are quite close to each other. The detail of the difference of the two is displayed in figure 3 for the noiseless situation.

6. Conclusion

We have derived the time-dependent solution of the multi-mode continuous variable state in the environment of amplification, amplitude damping and thermal noise. The evolution equation is expressed in the fashion of the characteristic function of canonical operators. The solution is analytically obtained for any multi-mode state. The result of the time-dependent state was applied in the analysis of the separability of tripartite three-mode Gaussian states when all three modes are identical in the initial preparation and in the later interaction with the environment through amplification, damping and thermal noise. The analytical expression of fully separable and biseparable conditions for the final tripartite three-mode Gaussian state is obtained when the initial Gaussian state is properly chosen. The set of initial Guassian states is still quite large although restricted; examples of those Gaussian states are vacuum state, tripartite three-mode pure entangled symmetric Gaussian state and tripartite squeezed thermal state [16]. The separability conditions are characterized by the ratio of the single-mode amplification parameter to the damping coefficient, the ratio of the inter-mode amplification parameter to the damping coefficient, and the thermal noise. The biseparable condition and the fully separable condition are very close to each other, make the domain of biseparable but not fully separable states quite small, while both the domains of the genuine entangled tripartite states and fully separable states are large enough. We use the same separability criterion as in [16]; the case that does not appear in [16] is that the residue states considered here may involve infinite large elements of the correlation matrix. Further analysis on the some non-symmetric tripartite Gaussian state can be found in [25].

Acknowledgments

Funding by the National Natural Science Foundation of China (under grant no. 10575092), the Zhejiang Province Natural Science Foundation (under grant no. RC104265) are gratefully acknowledged.

References

[1] Braunstein S L and Pati A K 2003 *Quantum Information with Continuous Variables* (Dordrecht: Kluwer)
[2] Braunstein S L and van Loock P 2005 *Rev. Mod. Phys.* **77** 513
[3] Adesso G and Illuminati F 2007 *J. Phys. A: Math. Gen.* **40** 7821
[4] Vaidman L 1994 *Phys. Rev.* A **49** 1473
Braunstein S L and Kimble H J 1998 *Phys. Rev. Lett.* **80** 869
[5] Furusawa A, Sorensen J L, Braunstein S L, Fuchs C A, Kimble H J and Polzik E S 1998 *Science* **282** 706
Zhang T C, Goh K W, Chou C W, Lodahl P and Kimble H J 2003 *Phys. Rev.* A **67** 033802
[6] Takei N, Yonezawa H, Aoki T and Furusawa A 2005 *Phys. Rev. Lett.* **94** 220502
[7] van Loock P and Braunstein S L 2000 *Phys. Rev. Lett.* **84** 3482
[8] Yonezawa H, Aoki T and Furusawa A 2004 *Nature* **431** 430
[9] Grosshans F and Grangier P 2002 *Phys. Rev. Lett.* **88** 057902
Grosshans F, Van Assche G, Wenger J, Brouri R, Cerf N J and Grangier P 2003 *Nature* **421** 238
Yuen H P and Kim A 1998 *Phys. Lett.* A **241** 135
Bencheikh K, Symul T, Jankovic A and Levenson J A 2001 *J. Mod. Opt.* **48** 1903
Grosshans F, Van Assche G, Wenger J, Brouri R, Cerf N J and Grangier P 2003 *Nature* **421** 238
[10] van Loock P and Braunstein S L 2000 *Phys. Rev.* A **61** 010302
[11] Jia X, Su X, Pan Q, Gao J, Xie C and Peng K 2004 *Phys. Rev. Lett.* **93** 250503
[12] Ban M 1999 *J. Opt.* B **1** L9
Braunstein S L and Kimble H J 2000 *Phys. Rev.* A **61** 042302
Li X, Pan Q, Jing J, Zhang J, Xie C and Peng K 2002 *Phys. Rev. Lett.* **88** 047904
Mizuno J, Wakui K, Furusawa A and Sasaki M 2005 *Phys. Rev.* A **71** 012304
[13] Julsgaard B, Sherson J, Cirac J I, Fiurasek J and Polzik E S 2004 *Nature* **432** 482
[14] Lloyd S and Braunstein S L 1999 *Phys. Rev. Lett.* **82** 1784
Ralph T C, Gilchrist A, Milburn G J, Munro W J and Glancy S 2003 *Phys. Rev.* A **68** 042319
[15] Giedke G, Kraus B, Lewenstein M and Cirac J I 2001 *Phys. Rev.* A **64** 052303
[16] Chen X Y 2005 *Phys. Lett.* A **335** 121
[17] Chen X Y 2006 *J. Phys. B: At. Mol. Opt. Phys.* **39** 4605
[18] Chen X Y, Jiang L Z and Chen J W 2007 *Phys. Lett.* A **366** 47
[19] Kinsler P and Drummond P D 1991 *Phys. Rev.* A **43** 6194
[20] Lindblad G 1976 *Commun. Math. Phys.* **48** 119
[21] Walls D and Milburn G 1994 *Quantum Optics* (Berlin: Springer)
[22] Leggett A J 2001 *Rev. Mod. Phys.* **73** 307
[23] Petz D 1990 *An Invitation to the Algebra of Canonical Commutation Relations* (Leuven: Leuven University Press)
[24] Perelomov A 1986 *Generalized Coherent States* (Berlin: Springer)
[25] Adesso G, Serafini A and Illuminati F 2007 *New J. Phys.* **9** 60